

Asymptotic Behaviour of Inhomogeneous String Cosmologies

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Abstract

The asymptotic behaviour at late times of inhomogeneous axion-dilaton cosmologies is investigated. The space-times considered here admit two abelian space-like Killing vectors. These space-times evolve towards an anisotropic universe containing gravitational radiation. Furthermore, a peeling-off behaviour of the Weyl tensor and the antisymmetric tensor field strength is found. The relation to the pre-big-bang scenario is briefly discussed.

1 Introduction

The low energy effective action of string theory provides cosmological models that might be applicable just below the string scale in the very early universe. In the pre-big-bang scenario the universe starts in a low curvature, low coupling regime and then enters a stage of dilaton driven kinetic inflation [1]. This picture has been further developed in [2]. Initially the universe consists of a bath of gravitational and dilatonic waves. Some of these collapse leading to the birth of a baby inflationary universe.

The aim here is to investigate the asymptotic future states of a large class of inhomogeneous string cosmologies. Late times in the usual general relativistic setting correspond to early times in the pre-big-bang picture. As will be seen, this provides further evidence that the Milne model is not a generic initial state for the pre-big-bang scenario. The issue of a generic initial state has also been investigated in [3] with the emphasis on orthogonal and tilted Bianchi string cosmologies. There it was found that a plane wave background is a likely attractor and not the Milne universe as was conjectured in [4]. Einstein-Rosen string cosmologies were discussed in [5] [6].

In general relativity the asymptotic evolution of gravitational radiation from a bounded source was investigated [7] and it was found that the Riemann tensor shows a “peeling off” behaviour, meaning that different components of the Weyl tensor evolve as different powers of the radial coordinate.

A cosmological peeling off theorem was formulated by Carmeli and Feinstein [8] by using earlier results of Stachel [9] in the case of cylindrical gravitational waves. Cosmological space-times of the Einstein-Rosen type can be obtained from these cylindrical gravitational wave space-times by interchanging the non-ignorable coordinates. The cosmological peeling off theorem describes the asymptotic future behaviour of inhomogeneous vacuum space-times admitting two abelian Killing vectors. The general form of the metric allows space-time homogeneity to be broken along one spatial direction, so in general solutions will depend on one spacelike and one timelike coordinate. However, this type of metric also encompasses the – by definition – spatially

homogeneous metrics of Bianchi type I-VII [10][8]. In [8] it was shown that the Weyl tensor shows a peeling off behaviour in terms of the timelike variable. At late times the dominant contribution comes from the Weyl scalar Ψ_4 which indicates that the space-time approaches at late times a plane wave space-time.

A similar peeling off behaviour will be found in the case of axion-dilaton string cosmology.

In the bosonic sector of the low energy effective action of a string theory general relativity coupled to two massless scalar fields is obtained in the so-called Einstein frame.

In four dimensions the action is given in the string frame by [11]

$$S = \int d^4x \sqrt{-g} e^{-\phi} (R + g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{12} H^{\alpha\beta\gamma} H_{\alpha\beta\gamma}) \quad (1)$$

where the antisymmetric tensor field strength $H_{\alpha\beta\gamma} = \partial_{[\alpha} B_{\beta\gamma]}$ is introduced.

Applying the conformal transformation

$$g_{\alpha\beta} \rightarrow e^{-\phi} g_{\alpha\beta}. \quad (2)$$

this can be written in the usual Einstein-Hilbert form.

The physical frame is the string frame since in this frame strings move along geodesics. However, since the Weyl tensor is conformally invariant it is justified to use the Einstein frame to find solutions and take advantage of the formalism developed for general relativity.

In the Einstein frame the equations of motion are given by

[11]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = {}^{(\phi)}T_{\mu\nu} + {}^{(H)}T_{\mu\nu} \quad (3)$$

$$\nabla_\mu [\exp(-2\phi) H^{\mu\nu\lambda}] = 0 \quad (4)$$

$$\square\phi + \frac{1}{6} e^{-2\phi} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0 \quad (5)$$

where

$${}^{(\phi)}T_{\mu\nu} = \frac{1}{2} (\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}) \quad (6)$$

$${}^{(H)}T_{\mu\nu} = \frac{1}{12} e^{-2\phi} \left(3 H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} - \frac{1}{2} g_{\mu\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \right) \quad (7)$$

$$(8)$$

Furthermore in four dimensions the antisymmetric tensor field strength can be written in terms of the scalar field, b , as follows

$$H^{\mu\nu\lambda} = e^{2\phi} \epsilon^{\rho\mu\nu\lambda} b_{,\rho}. \quad (9)$$

2 Equations of motion for axion and dilaton

Metrics admitting two spacelike abelian Killing vectors can be written in the following general form [12]

$$ds^2 = 2e^{-M} du dv - \frac{2e^{-U}}{Z + \bar{Z}} (dx + iZ dy) (dx - i\bar{Z} dy) \quad (10)$$

M and U are real functions depending on the null variables u and v . Z is a complex function depending on u and v . Complex conjugation is denoted by a bar.

It is convenient to introduce a null tetrad with two real null vectors, \mathbf{l} , \mathbf{n} , and two complex null vectors which are conjugates of one another, \mathbf{m} , $\bar{\mathbf{m}}$.

They satisfy the following relations [12][13]

$$l_\mu n^\mu = 1 \quad m_\mu \bar{m}^\mu = -1 \quad (11)$$

and the completeness relation

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \quad (12)$$

Useful formulae for quantities in the Newman-Penrose formalism are given in the appendix.

Including the axion b and dilaton ϕ and using Einstein's equations the components of the Ricci tensor can be determined. The energy momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{2} [\phi_{,\mu} \phi_{,\nu} + e^{2\phi} b_{,\mu} b_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\phi_{,\alpha} \phi_{,\beta} + e^{2\phi} b_{,\alpha} b_{,\beta})]. \quad (13)$$

The following equations of motion are obtained

$$2U_{uu} - (U_u)^2 + 2M_u U_u - 4 \frac{Z_u \bar{Z}_u}{(Z + \bar{Z})^2} = \phi_u^2 + e^{2\phi} b_u^2 \quad (14)$$

$$2U_{vv} - (U_v)^2 + 2M_v U_v - 4 \frac{Z_v \bar{Z}_v}{(Z + \bar{Z})^2} = \phi_v^2 + e^{2\phi} b_v^2 \quad (15)$$

$$2Z_{uv} - U_u Z_v - U_v Z_u - 4 \frac{Z_u Z_v}{Z + \bar{Z}} = 0 \quad (16)$$

$$2M_{uv} + U_u U_v - 2 \frac{Z_u \bar{Z}_v + \bar{Z}_u Z_v}{(Z + \bar{Z})^2} = \phi_u \phi_v + e^{2\phi} b_u b_v \quad (17)$$

$$U_{uv} = U_u U_v \quad (18)$$

$$2\phi_{uv} - U_u \phi_v - U_v \phi_u - 2e^{2\phi} b_u b_v = 0 \quad (19)$$

$$2b_{uv} - U_u b_v - U_v b_u + 2\phi_u b_v + 2\phi_v b_u = 0 \quad (20)$$

Equation (17) is an integrability condition for equations (14) and (15). The general solution of equation (18) is given by [12]

$$e^{-U} = f(u) + g(v) \quad (21)$$

where f and g are arbitrary functions of their arguments. This can be used to write the equations of motion in a more compact way by using f and g as coordinates.

The equations can be simplified further by introducing a new function S , defined as follows [12]

$$e^{-M} = \frac{f'g'}{\sqrt{f+g}} e^{-S} \quad (22)$$

where the prime denotes a derivative w.r.t. to u (v) for f (g).

So this reduces the system of equations (14)–(20) to the following

$$2Z_{fg} + \frac{1}{f+g} (Z_f + Z_g) - 4 \frac{Z_f Z_g}{Z + \bar{Z}} = 0 \quad (23)$$

$$2\phi_{fg} + \frac{1}{f+g} (\phi_f + \phi_g) - 2e^{2\phi} b_f b_g = 0 \quad (24)$$

$$2b_{fg} + \frac{1}{f+g} (b_f + b_g) + 2\phi_f b_g + 2\phi_g b_f = 0 \quad (25)$$

$$S_f = -2(f+g) \left(\frac{Z_f \bar{Z}_f}{(Z + \bar{Z})^2} + \frac{1}{4} (\phi_f^2 + e^{2\phi} b_f^2) \right) \quad (26)$$

$$S_g = -2(f+g) \left(\frac{Z_g \bar{Z}_g}{(Z + \bar{Z})^2} + \frac{1}{4} (\phi_g^2 + e^{2\phi} b_g^2) \right). \quad (27)$$

Therefore the components of the Weyl tensor are given by

$$\Psi_0 = \frac{e^M}{(Z + \bar{Z})^2} (g')^2 \left[(Z + \bar{Z}) \left(\bar{Z}_{gg} + \left[\frac{3}{2} \frac{1}{f+g} - 2(f+g) \left[\frac{Z_g \bar{Z}_g}{(Z + \bar{Z})^2} + \frac{1}{4} (\phi_g^2 + e^{2\phi} b_g^2) \right] \right) \bar{Z}_g \right) - 2(\bar{Z}_g)^2 \right] \quad (28)$$

$$\Psi_2 = -\frac{e^M}{12} f' g' \left[\frac{3}{(f+g)^2} - 12 \frac{Z_f \bar{Z}_g}{(Z + \bar{Z})^2} - (\phi_f \phi_g + e^{2\phi} b_f b_g) \right] \quad (29)$$

$$\Psi_4 = \frac{e^M}{(Z + \bar{Z})^2} (f')^2 \left[(Z + \bar{Z}) \left(Z_{ff} + \left[\frac{3}{2} \frac{1}{f+g} - 2(f+g) \left[\frac{Z_f \bar{Z}_f}{(Z + \bar{Z})^2} + \frac{1}{4} (\phi_f^2 + e^{2\phi} b_f^2) \right] \right) Z_f \right) - 2(Z_f)^2 \right] \quad (30)$$

3 Exact solutions

In this section several exact solutions are discussed. These are of interest since they will already show some of the behaviour of the more general space-times which will be discussed in the next section. Introducing two new variables,

$$f = t - z \quad g = t + z \quad (31)$$

and the Ernst potential Z may be conveniently written as

$$Z = \zeta + i\omega \quad (32)$$

where ζ and ω are real functions. When the imaginary part of the Ernst potential vanishes, the two commuting Killing vectors become hypersurface orthogonal. In this case the metric is globally diagonalizable.

In the special case of a diagonal and homogeneous ansatz equation (23) reads

$$\zeta(\zeta_{tt} + t^{-1}\zeta_t) - \zeta_t^2 = 0 \quad (33)$$

which is solved by

$$\zeta = at^c \quad (34)$$

with a and c constants.

The equations for the dilaton and axion are given by

$$\phi_{tt} + t^{-1}\phi_t - e^{2\phi}b_t^2 = 0 \quad (35)$$

$$b_{tt} + t^{-1}b_t + 2\phi_t b_t = 0 \quad (36)$$

which is solved by [14] [5]

$$e^\phi = \cosh(N\xi) + \sqrt{1 - \frac{B^2}{N^2}} \sinh(N\xi) \quad (37)$$

$$b(\xi) = \frac{N \sinh(N\xi) + \sqrt{1 - \frac{B^2}{N^2}} \cosh(N\xi)}{B \cosh(N\xi) + \sqrt{1 - \frac{B^2}{N^2}} \sinh(N\xi)} \quad (38)$$

where $dt = t d\xi$. This solution can actually be obtained by an $SL(2; \mathbb{R})$ transformation from a pure dilaton solution.

The components of the Weyl tensor are given by

$$\Psi_0 = \frac{e^M}{16} (g')^2 c [1 - c^2 - N^2] t^{-2} \quad (39)$$

$$\Psi_2 = -\frac{e^M}{48} f' g' [3(1 - c^2) - N^2] t^{-2} \quad (40)$$

$$\Psi_4 = \frac{e^M}{16} (f')^2 c [1 - c^2 - N^2] t^{-2} \quad (41)$$

Using (34) as ansatz for ζ and solving (23) for ω results in an inhomogeneous solution for ω [9]. In this case (23) yields the following two equations

$$\zeta(\zeta_{tt} + t^{-1}\zeta_t) - (\zeta_t^2 - \omega_t^2) - \omega_z^2 = 0 \quad (42)$$

$$\zeta(\omega_{tt} + t^{-1}\omega_t - \omega_{zz}) - 2\omega_t \zeta_t = 0. \quad (43)$$

For the special value $c = \frac{1}{2}$ this has the solution

$$\zeta = at^{\frac{1}{2}} \quad (44)$$

$$\omega = \Theta(\kappa)F(t-z) + (1-\Theta(\kappa))G(t+z) \quad (45)$$

where $\Theta(\kappa)$ is the step function ($\Theta(\kappa) = 0$ for $\kappa \leq 0$; $\Theta(\kappa) = 1$ for $\kappa > 0$) and κ is an arbitrary real parameter. Note that ω can either include F or G , but not both.

Another solution in this class could be easily generated by applying a mirror transformation. Mirror symmetries in the context of metrics with two abelian Killing vectors were discussed in [15]. The effect of such a transformation in the class of backgrounds considered here is to exchange the background metric and the axion-dilaton content. Applying such a transformation to the solution above (37), (38), (44), (45) results in a solution with a homogeneous dilaton and an inhomogeneous, oscillating axion similar to the solution found in [5]. Since the solution for the axion and dilaton can be obtained by an $SL(2; \mathbb{R})$ transformation the space-time metric in the mirror image is not genuinely diagonal.

The components of the Weyl tensor are given by

$$\begin{aligned} \Psi_0 = & e^M (g')^2 \left[\frac{1}{32} \left(\frac{3}{4} - N^2 \right) t^{-2} + \frac{3}{8a^2} (1 - \Theta(\kappa))^2 [G'(g)]^2 t^{-1} \right. \\ & - \frac{i}{8a} \left(\frac{3}{4} - N^2 \right) (1 - \Theta(\kappa)) G'(g) t^{-\frac{3}{2}} + \frac{i}{2a^3} (1 - \Theta(\kappa))^3 [G'(g)]^3 t^{-\frac{1}{2}} \\ & \left. - \frac{i}{2a} (1 - \Theta(\kappa)) G''(g) t^{-\frac{1}{2}} \right] \end{aligned} \quad (46)$$

$$\begin{aligned} \Psi_2 = & -\frac{e^M}{48} f' g' \left[\left(\frac{9}{4} - N^2 \right) t^{-2} - \frac{12}{a^2} (1 - \Theta(\kappa)) \Theta(\kappa) G'(g) F'(f) t^{-1} \right. \\ & \left. - i \frac{3}{a} [\Theta(\kappa) F'(f) - (1 - \Theta(\kappa)) G'(g)] t^{-\frac{3}{2}} \right] \end{aligned} \quad (47)$$

$$\begin{aligned} \Psi_4 = & e^M (f')^2 \left[\frac{1}{32} \left(\frac{3}{4} - N^2 \right) t^{-2} + \frac{3}{8a^2} \Theta(\kappa)^2 [F'(f)]^2 t^{-1} \right. \\ & + \frac{i}{8a} \left(\frac{3}{4} - N^2 \right) \Theta(\kappa) F'(f) t^{-\frac{3}{2}} - \frac{i}{2a^3} \Theta(\kappa)^3 [F'(f)]^3 t^{-\frac{1}{2}} \\ & \left. + \frac{i}{2a} \Theta(\kappa) F''(f) t^{-\frac{1}{2}} \right]. \end{aligned} \quad (48)$$

For this solution the equations for S , (26), (27), have the form

$$S_f = -\frac{1}{2} \left(\frac{1}{4} + N^2 \right) \frac{1}{f+g} - \frac{1}{a^2} [\omega_f]^2 \quad (49)$$

$$S_g = -\frac{1}{2} \left(\frac{1}{4} + N^2 \right) \frac{1}{f+g} - \frac{1}{a^2} [\omega_g]^2. \quad (50)$$

This system of equations can be solved by introducing a new function $\Omega(f, g)$ such that

$$\Omega_f = [\omega_f]^2 \quad \Omega_g = [\omega_g]^2 \quad (51)$$

which implies, using that $\omega_{fg} = 0$,

$$\Omega_{fg} = 0.$$

This equation has the general solution

$$\Omega(f, g) = \Omega_1(f) + \Omega_2(g)$$

and hence using (51)

$$\Omega(f, g) = \int df(\omega_f)^2 + \int dg(\omega_g)^2. \quad (52)$$

Thus S is given by

$$S = -\frac{1}{2}\left(\frac{1}{4} + N^2\right) \ln(f + g) - \frac{1}{a^2} \Omega + \ln k \quad (53)$$

where k is an arbitrary constant.

With equation (22) the factor e^M in the expressions for the components of the Weyl tensor is given by

$$e^M = k(f'g')^{-1} t^{\frac{3}{8} - \frac{1}{2}N^2} e^{-\frac{1}{a^2}\Omega}. \quad (54)$$

For $\Theta(\kappa) = 1$ and $f = t - z$ fixed, then the components of the Weyl tensor behave as $\Psi_0 \sim e^M (g')^2 t^{-2}$, $\Psi_2 \sim e^M f' g' t^{-\frac{3}{2}}$, $\Psi_4 \sim e^M (f')^2 t^{-\frac{1}{2}}$ as $t \rightarrow \infty$.

A similar behaviour is found for $\Theta(\kappa) = 0$ and $g = t + z$ fixed in the limit $t \rightarrow \infty$. In this case the behaviour of Ψ_0 and Ψ_4 is exchanged. In both cases a plane wave space-time is approached. However, the plane waves are in opposing directions.

Furthermore, independent of $\Theta(\kappa)$, the pre-factor e^M is given by $\mathcal{O}(t^{\frac{3}{8} - \frac{1}{2}N^2})$

Finally, one more class of exact solutions will be discussed. The metric is again assumed to be diagonal. With $\zeta = e^{-V}$ the Ernst equation (23) implies

$$V_{tt} + \frac{1}{t}V_t - V_{zz} = 0. \quad (55)$$

This is solved by [12]

$$\begin{aligned} V = & -a \ln t + L_1 \{A_\omega \cos[\omega(z + \alpha_\omega)] J_0(\omega t)\} + L_2 \{B_\omega \cos[\omega(z + \beta_\omega)] Y_0(\omega t)\} \\ & + \sum_i d_i \cosh^{-1} \left(\frac{z + c_i}{t} \right) \end{aligned} \quad (56)$$

where $L_1\{\}$ and $L_2\{\}$ are arbitrary combinations of the terms in curly brackets. J_0 and Y_0 are Bessel functions of the first and second kind. A_ω , B_ω , α_ω and β_ω are arbitrary constants.

The first term on its own describes a Kasner solution. The terms containing the Bessel functions will be investigated in the next section. The last term introduces the so-called gravitosoliton contribution.

In the simplest case, the gravitosoliton contribution contains just one term and the constant c is set to zero. The constant pre-factor is d .

Neglecting the contributions from the Bessel functions solutions by Wainwright et al. are recovered [16]. These are related to the Kasner metrics by a soliton transformation [17] [18].

Furthermore, for $d^2 = a^2 + 3$ the metrics are the Ellis-MacCallum solutions of Bianchi type III, V and VI [19] [20] [18]. Coordinates \tilde{t} , \tilde{z} adapted to spatial homogeneity are given by [20][18]

$$t = e^{-2m\tilde{z}} \sinh 2m\tilde{t} \quad z = e^{-2m\tilde{z}} \cosh 2m\tilde{t}.$$

Depending on the parameter a the models are of Bianchi type III, V or VI.

As was pointed out by Carmeli et al. [20], in general these space-times are not only singular at $\tilde{t} = 0$ but also at $\tilde{z} \rightarrow \pm\infty$. This behaviour depends on the parameters a and d . Only the Bianchi models are singularity free at $\tilde{z} \rightarrow \pm\infty$. If one is interested in a cosmological interpretation this seems to confine solutions containing a gravitosolitic contribution to those which are spatially homogeneous.

Orthogonal Bianchi VI_h models approach a plane wave space-time at future timelike and null infinity. The other two approach flat space-time [21] [3].

4 Asymptotic behaviour of the Weyl tensor and antisymmetric tensor field strength

The Petrov type of the Weyl tensor describes the number of its distinct principal null directions. It is determined by the roots of the following equation [13]

$$\Psi_4 b^4 + 4\Psi_3 b^3 + 6\Psi_2 b^2 + 4\Psi_1 b + \Psi_0 = 0 \quad (57)$$

In the case of metrics of type (10) the complex Weyl scalars Ψ_1 and Ψ_3 vanish. Therefore there are either four distinct or two double roots. So the space-time is of type I or of type D.

In order to find the Petrov type of a space-time an algorithm developed in [22] can be applied. In the case of the metric considered here, the criterion for the space-time being of type D is [22] [18]

$$I^3 = 27J^2 \quad (58)$$

where $I \equiv \Psi_0\Psi_4 + 3\Psi_2^2$ and $J \equiv \Psi_2(\Psi_0\Psi_4 - \Psi_2^2)$.

In general the space-time is of type I. However, in the diagonal solution of the last section there exist special values for c such that the Weyl tensor is of type D. These are given by $c = \pm 1$, $c = \pm\sqrt{1 - N^2}$ and $c = \pm(1 \pm \sqrt{4 - N^2})$.

In cosmology one is interested in the evolution of a space-time with time. Therefore future timelike and future null infinity will be discussed here.

Following [9] Z will be taken of the form

$$Z = te^{-2\psi}(1 + i2\chi) \quad (59)$$

where ψ and χ are functions of t and f . In this case ψ and χ enter more symmetric in the Ernst equation and show the same asymptotic behaviour, which in both cases will be $\mathcal{O}(t^{-\frac{1}{2}})$ for ψ and χ . This asymptotic behaviour show the dilaton and axion as well. This is not surprising since the linear terms in the Ernst equation, the equations for the axion and the dilaton have the same structure (cf (23)-(25)).

At future timelike infinity ($t \rightarrow \infty$, z fixed) the Weyl scalars behave as

$$\begin{aligned}
\Psi_0 &\sim e^M \mathcal{O}(t^{-\frac{1}{2}}) \\
\Psi_2 &\sim e^M \mathcal{O}(t^{-1}) \\
\Psi_4 &\sim e^M \mathcal{O}(t^{-\frac{1}{2}}).
\end{aligned} \tag{60}$$

This shows that the space-time does not approach a Kasner model (cf. equations (39)–(41)). This can be compared with the future time infinity of the Doroshkevich, Zeldovich, Novikov (DZN) [23] [3] model for which the components of the Weyl tensor are given by $\Psi_0 \sim e^M t^{-1}$, $\Psi_2 \sim e^M t^{-2}$, $\Psi_4 \sim e^M t^{-1}$ and the metric is $ds^2 = e^{2t}(dt^2 - dx^2) - t^{d+1}dy^2 - t^{1-d}dz^2$. So these models show a similar behaviour of the Weyl scalars at future time infinity. This suggests that the nondiagonal Einstein-Rosen space-times approach an anisotropic model similar to the DZN model. However, they certainly do not approach the Milne model for which $\Psi_0 \sim \Psi_2 \sim \Psi_4 \sim \mathcal{O}(t^{-2})$. This can be compared with the asymptotic behaviour of diagonal Einstein-Rosen axion-dilaton cosmologies. In [5] exact solutions were found which approach a DZN universe containing axionic, dilatonic and gravitational waves at late times. A similar behaviour is also found in electromagnetic Gowdy universes [24].

To investigate the behaviour of the Weyl tensor at future null infinity one can use one null coordinate, say, f and the timelike coordinate t .

In these variables the Ernst equation (23) becomes

$$Z_{tf} + \frac{1}{2}Z_{tt} + \frac{1}{2t}(Z_t + Z_f) = 2(Z + \bar{Z})^{-1} \left[Z_t Z_f + \frac{1}{2}Z_t^2 \right] \tag{61}$$

The linear part of the real and imaginary parts of (61) yield to two cylindrical wave equations for ψ and χ , respectively. These can be solved by Bessel functions (cf (56)). Their asymptotic behaviour in the limit $t \rightarrow \infty$, f fixed, imply that both ψ and χ behave as $a(f)t^{-\frac{1}{2}}$, if gravitonsoliton contributions are neglected. These were discussed for diagonal metrics in the last section. They seem to lead to a cosmological model only in the spatially homogeneous case.

In terms of the coordinates t and f the non-vanishing components of the Weyl tensor are given by

$$\begin{aligned}
\Psi_0 &= \frac{e^M}{4}(g')^2 \left[\frac{\bar{Z}_{tt}}{Z + \bar{Z}} + \left(\frac{3}{2t} - 2t \left[\frac{Z_t \bar{Z}_t}{(Z + \bar{Z})^2} + \frac{1}{4}(\phi_t^2 + e^{2\phi} b_t^2) \right] \right) \frac{\bar{Z}_t}{Z + \bar{Z}} \right. \\
&\quad \left. - 2 \frac{\bar{Z}_t^2}{(Z + \bar{Z})^2} \right]
\end{aligned} \tag{62}$$

$$\Psi_2 = -\frac{e^M}{24} f' g' \left[\frac{3}{2t^2} - 12 \frac{(Z_f + \frac{1}{2}Z_t)\bar{Z}_t}{(Z + \bar{Z})^2} - (\phi_f + \frac{1}{2}\phi_t)\phi_t - e^{2\phi}(b_f + \frac{1}{2}b_t)b_t \right] \tag{63}$$

$$\begin{aligned}
\Psi_4 &= e^M (f')^2 \left[(Z + \bar{Z})^{-1} (Z_{ff} + Z_{ft} + \frac{1}{4}Z_{tt}) \right. \\
&\quad + \left[\frac{3}{4t} - 4t \left(\frac{(Z_f + \frac{1}{2}Z_t)(\bar{Z}_f + \frac{1}{2}\bar{Z}_t)}{(Z + \bar{Z})^2} + \frac{1}{4} \left[(\phi_f + \frac{1}{2}\phi_t)^2 + e^{2\phi}(b_f + \frac{1}{2}b_t)^2 \right] \right) \right] \frac{Z_f + \frac{1}{2}Z_t}{Z + \bar{Z}} \\
&\quad \left. - 2 \frac{(Z_f + \frac{1}{2}Z_t)^2}{(Z + \bar{Z})^2} \right]
\end{aligned} \tag{64}$$

With Z given by (59) and the asymptotic behaviour of ψ and χ the components of the Weyl tensor behave as follows

$$\Psi_0 \sim \frac{e^M}{4}(g')^2[\mathcal{O}(t^{-2})] \quad (65)$$

$$\Psi_2 \sim -\frac{e^M}{24}f'g'[\mathcal{O}(t^{-\frac{3}{2}})] \quad (66)$$

$$\Psi_4 \sim e^M(f')^2[\mathcal{O}(t^{-\frac{1}{2}})] \quad (67)$$

So as $t \rightarrow \infty$ the component Ψ_4 dominates and the space-time approaches a plane wave space-time at future null infinity.

This behaviour was already found in the exact non-diagonal solution of the last section. There the two cases of an ingoing and outgoing wave had been stated explicitly. The same applies here as well. Interchanging the null coordinates implies an interchange of the behaviour of Ψ_0 and Ψ_4 . Hence outgoing and ingoing waves are exchanged.

The equations for dilaton and axion are given by

$$\phi_{ft} + \frac{1}{2}\phi_{tt} + \frac{1}{2t}[\phi_f + \phi_t] - e^{2\phi}b_t[b_f + \frac{1}{2}b_t] = 0 \quad (68)$$

$$b_{ft} + \frac{1}{2}b_{tt} + \frac{1}{2t}[b_f + b_t] + b_t[\phi_f + \frac{1}{2}\phi_t] + \phi_t[b_f + \frac{1}{2}b_t] = 0. \quad (69)$$

Hence, in the limit $t \rightarrow \infty$ the behaviour of the dilaton and axion is described by,

$$\phi \sim \mathcal{O}(t^{-\frac{1}{2}}) \quad (70)$$

$$b \sim \mathcal{O}(t^{-\frac{1}{2}}). \quad (71)$$

This leads to a peeling-off behaviour of the components of the antisymmetric tensor field strength H . This is similar to the Einstein-Maxwell case where a peeling-off behaviour for the components of the Maxwell tensor is found in addition to that of the Weyl tensor [25].

Equation (9) implies that the non-vanishing components of the antisymmetric tensor field strength in the null tetrad are given by

$$H_{(1)(3)(4)} = -i\frac{1}{2}e^{2\phi}e^{\frac{M}{2}}f'(b_f + \frac{1}{2}b_t) \quad (72)$$

$$H_{(2)(3)(4)} = i\frac{1}{2}e^{2\phi}e^{\frac{M}{2}}g'b_t. \quad (73)$$

This leads to the asymptotic behaviour,

$$H_{(2)(3)(4)} \sim e^{2\phi}e^{\frac{M}{2}}\mathcal{O}(t^{-\frac{3}{2}}) \quad (74)$$

$$H_{(1)(3)(4)} \sim e^{2\phi}e^{\frac{M}{2}}\mathcal{O}(t^{-\frac{1}{2}}). \quad (75)$$

5 Discussion

The asymptotic behaviour of a large class of inhomogeneous axion-dilaton cosmologies has been investigated. At future time infinity the Weyl scalars approach those of an anisotropic space-time. At future null infinity the radiative component of the Weyl scalars dominates. This might be interpreted as that this class of inhomogeneous string cosmologies approaches an expanding anisotropic background on which gravitational waves propagate. Axion and dilaton become negligible at late times.

A peeling-off behaviour of the components of the Weyl tensor has been found similar to that in the case of vacuum general relativity. The components of the antisymmetric tensor field strength also show a peeling-off behaviour. This is analog to the asymptotic behaviour of the Maxwell tensor in the Einstein-Maxwell case.

The future asymptotic state in general relativity can be interpreted as the past asymptotic state for the pre-big-bang scenario. In the class of inhomogeneous metrics considered here, this means that an anisotropic model and not the Milne universe is a likely past attractor.

Furthermore, the type of metric which was studied here might also be relevant to the recently proposed pre-big-bang bubble picture. Together with appropriate boundary conditions metrics of the general form (10) also describe colliding plane wave space-times. This might be used to find a “dynamical” description of the formation of pre-big-bang bubbles [2].

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Appendix: Components of the Weyl and Ricci tensors

Here all the non-vanishing components of the Weyl tensor and the Ricci tensor in the Newman-Penrose formalism are given. The signature of the metric is $(+ - - -)$ and Einstein’s equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -T_{\mu\nu}. \quad (76)$$

The tetrad metric to raise and lower tetrad indices is given by

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (77)$$

Tetrad indices are enclosed in brackets, which will be omitted if it is unambiguous. Tetrad indices run from 1 to 4.

For a space-time with metric (10) a null tetrad basis is provided by [12]

$$e_{(1)} = e^{(2)} = l_\mu = e^{-\frac{M}{2}} \left\{ \begin{matrix} 1 & 0 & 0 & 0 \end{matrix} \right\}$$

$$\begin{aligned}
e_{(2)} &= e^{(1)} = n_\mu = e^{-\frac{M}{2}} \left\{ \begin{matrix} 0 & 1 & 0 & 0 \end{matrix} \right\} \\
-e_{(3)} &= e^{(4)} = -m_\mu = e^{-\frac{U}{2}} (Z + \bar{Z})^{-\frac{1}{2}} \left\{ \begin{matrix} 0 & 0 & 1 & -i\bar{Z} \end{matrix} \right\} \\
-e_{(4)} &= e^{(3)} = -\bar{m}_\mu = e^{-\frac{U}{2}} (Z + \bar{Z})^{-\frac{1}{2}} \left\{ \begin{matrix} 0 & 0 & 1 & iZ \end{matrix} \right\}
\end{aligned} \tag{78}$$

and the corresponding contravariant components are given by

$$\begin{aligned}
l^\mu &= e^{\frac{M}{2}} \left\{ \begin{matrix} 0 & 1 & 0 & 0 \end{matrix} \right\} \\
n^\mu &= e^{\frac{M}{2}} \left\{ \begin{matrix} 1 & 0 & 0 & 0 \end{matrix} \right\} \\
m^\mu &= e^{\frac{U}{2}} (Z + \bar{Z})^{-\frac{1}{2}} \left\{ \begin{matrix} 0 & 0 & \bar{Z} & -i \end{matrix} \right\} \\
\bar{m}^\mu &= e^{\frac{U}{2}} (Z + \bar{Z})^{-\frac{1}{2}} \left\{ \begin{matrix} 0 & 0 & Z & i \end{matrix} \right\}.
\end{aligned} \tag{79}$$

where $(x^0, x^1, x^2, x^3) \equiv (u, v, x, y)$.

The Newman-Penrose spin coefficients have been calculated using Maple [26],

$$\begin{aligned}
\lambda &= -e^{\frac{M}{2}} \frac{Z_u}{Z + \bar{Z}} \\
\mu &= -\frac{1}{2} e^{\frac{M}{2}} U_u \\
\rho &= \frac{1}{2} e^{\frac{M}{2}} U_v \\
\sigma &= e^{\frac{M}{2}} \frac{\bar{Z}_v}{Z + \bar{Z}} \\
\gamma &= \frac{1}{4} e^{\frac{M}{2}} \left[M_u - \frac{Z_u - \bar{Z}_u}{Z + \bar{Z}} \right] \\
\epsilon &= -\frac{1}{4} e^{\frac{M}{2}} \left[M_v + \frac{Z_v - \bar{Z}_v}{Z + \bar{Z}} \right] \\
\nu &= \pi = \tau = \kappa = \alpha = \beta = 0
\end{aligned} \tag{80}$$

F_u denotes the partial derivative $\frac{\partial F}{\partial u}$.

In the Newman-Penrose formalism the components of the Weyl tensor are given by five complex scalars [13]

$$\begin{aligned}
\Psi_0 &= -C_{(1)(3)(1)(3)} = -C_{\mu\nu\lambda\kappa} l^\mu m^\nu l^\lambda m^\kappa \\
\Psi_1 &= -C_{(1)(2)(1)(3)} = -C_{\mu\nu\lambda\kappa} l^\mu n^\nu l^\lambda m^\kappa \\
\Psi_2 &= -C_{(1)(3)(4)(2)} = -C_{\mu\nu\lambda\kappa} l^\mu m^\nu \bar{m}^\lambda n^\kappa \\
\Psi_3 &= -C_{(1)(2)(4)(2)} = -C_{\mu\nu\lambda\kappa} l^\mu n^\nu \bar{m}^\lambda n^\kappa \\
\Psi_4 &= -C_{(2)(4)(2)(4)} = -C_{\mu\nu\lambda\kappa} n^\mu \bar{m}^\nu n^\lambda \bar{m}^\kappa
\end{aligned} \tag{81}$$

and the components of the Ricci tensor are denoted by the following four real and three complex scalars [13]

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{(1)(1)} & \Phi_{22} &= -\frac{1}{2}R_{(2)(2)} \\
\Phi_{02} &= -\frac{1}{2}R_{(3)(3)} & \Phi_{20} &= -\frac{1}{2}R_{(4)(4)} \\
\Phi_{11} &= -\frac{1}{4}(R_{(1)(2)} + R_{(3)(4)}) & \Phi_{01} &= -\frac{1}{2}R_{(1)(3)} \\
\Lambda = \frac{1}{24}R &= \frac{1}{12}(R_{(1)(2)} - R_{(3)(4)}) & \Phi_{10} &= -\frac{1}{2}R_{(1)(4)} \\
\Phi_{12} &= -\frac{1}{2}R_{(2)(3)} & \Phi_{21} &= -\frac{1}{2}R_{(2)(4)}
\end{aligned}$$

Using the relations given in [13] the components of the Weyl and the Ricci tensor are derived. The components of the Weyl tensor are given by

$$\begin{aligned}
\Psi_0 &= \frac{e^M}{(Z + \bar{Z})^2} \left[(Z + \bar{Z})(\bar{Z}_{vv} + M_v \bar{Z}_v - U_v \bar{Z}_v) - 2(\bar{Z}_v)^2 \right] \\
\Psi_2 &= -\frac{1}{4}e^M \left(2U_{uv} - U_u U_v - 4\frac{Z_u \bar{Z}_v}{(Z + \bar{Z})^2} \right) - 2\Lambda \\
\Psi_4 &= \frac{e^M}{(Z + \bar{Z})^2} \left[(Z + \bar{Z})(Z_{uu} + M_u Z_u - U_u Z_u) - 2(Z_u)^2 \right] \\
\Psi_1 &= \Psi_3 = 0
\end{aligned} \tag{82}$$

and the non-vanishing components of the Ricci tensor are

$$\begin{aligned}
\Phi_{00} &= \frac{e^M}{4} \left[2U_{vv} - (U_v)^2 + 2M_v U_v - 4\frac{Z_v \bar{Z}_v}{(Z + \bar{Z})^2} \right] \\
\Phi_{02} &= -\frac{1}{2}\frac{e^M}{(Z + \bar{Z})^2} \left[(Z + \bar{Z})(2\bar{Z}_{uv} - U_u \bar{Z}_v - U_v \bar{Z}_u) - 4\bar{Z}_u \bar{Z}_v \right] \\
\Phi_{11} &= \frac{e^M}{8} \left[2M_{uv} + U_u U_v - 2\frac{Z_u \bar{Z}_v + \bar{Z}_u Z_v}{(Z + \bar{Z})^2} \right] \\
\Phi_{22} &= \frac{e^M}{4} \left[2U_{uu} - (U_u)^2 + 2U_u M_u - 4\frac{Z_u \bar{Z}_u}{(Z + \bar{Z})^2} \right] \\
\Lambda &= -\frac{e^M}{24} \left[2M_{uv} + 4U_{uv} - 3U_u U_v - 2\frac{Z_u \bar{Z}_v + \bar{Z}_u Z_v}{(Z + \bar{Z})^2} \right].
\end{aligned} \tag{83}$$

The non-vanishing components and the trace in a space-time with metric (10) are given by

$$\begin{aligned}
T_{00} &= \frac{1}{2}[\phi_u^2 + e^{2\phi} b_u^2] & T_{11} &= \frac{1}{2}[\phi_v^2 + e^{2\phi} b_v^2] \\
T_{22} &= \frac{e^{M-U}}{Z + \bar{Z}}[\phi_u \phi_v + e^{2\phi} b_u b_v] & T_{23} &= \frac{i}{2}e^{M-U} \frac{Z - \bar{Z}}{Z + \bar{Z}}[\phi_u \phi_v + e^{2\phi} b_u b_v] \\
T_{33} &= e^{M-U} \frac{Z \bar{Z}}{Z + \bar{Z}}[\phi_u \phi_v + e^{2\phi} b_u b_v] & T &= -e^M[\phi_u \phi_v + e^{2\phi} b_u b_v]
\end{aligned} \tag{84}$$

and using Einstein's equations the only non-vanishing components of the Ricci tensor are as follows

$$R_{00} = -\frac{1}{2}[\phi_u^2 + e^{2\phi}b_u^2] \quad R_{01} = -\frac{1}{2}[\phi_u\phi_v + e^{2\phi}b_ub_v] \quad R_{11} = -\frac{1}{2}[\phi_v^2 + e^{2\phi}b_v^2]. \quad (85)$$

So the non-vanishing Newman-Penrose scalars Φ_{ab} and Λ are given by

$$\begin{aligned} \Phi_{00} &= \frac{e^M}{4}[\phi_v^2 + e^{2\phi}b_v^2] & \Phi_{11} &= \frac{e^M}{8}[\phi_u\phi_v + e^{2\phi}b_ub_v] \\ \Phi_{22} &= \frac{e^M}{4}[\phi_u^2 + e^{2\phi}b_u^2] & \Lambda &= -\frac{e^M}{24}[\phi_u\phi_v + e^{2\phi}b_ub_v]. \end{aligned} \quad (86)$$

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